NPS-53-89-004

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A NOTE ON AN INVERSE EIGENPROBLEM
FOR BAND MATRICES

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November 1988

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Prepared for: Naval Postgraduate School and the
National Science Foundation, Washington
D.C. 20550

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This report was prepared in conjunction with research conducted for the National Science Foundation and for the Naval Postgraduate School Research Council and funded by the Naval Postgraduate School Research Council.

Prepared by:

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited
2b DECLASSIFICATION/DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S) NPS-53-89-004	5 MONITORING ORGANIZATION REPORT NUMBER(S) NPS-53-89-004
6a NAME OF PERFORMING ORGANIZATION Naval Postgraduate School 6b OFFICE SYMBOL (If applicable) 53	7a NAME OF MONITORING ORGANIZATION Naval Postgraduate School and the National Science Foundation
6c ADDRESS (City, State, and ZIP Code) Monterey, CA 93943	7b ADDRESS (City, State, and ZIP Code) Monterey, CA 93943 and Washington, D.C.
8a NAME OF FUNDING/SPONSORING ORGANIZATION Naval Postgraduate School 8b OFFICE SYMBOL (If applicable) 53	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER O&MN, direct funding
8c ADDRESS (City, State, and ZIP Code) Monterey, CA 93943	10 SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO / PROJECT NO / TASK NO / WORK UNIT ACCESSION NO

11 TITLE (Include Security Classification)
A Note On An Inverse Eigenproblem For Band Matrices

12 PERSONAL AUTHOR(S)
William B. Gragg and Gregory S. Ammar

13a TYPE OF REPORT	13b TIME COVERED	14 DATE OF REPORT (Year, Month, Day)	15 PAGE COUNT
Technical Report	FROM 10/1/87 TO 9/30/88	1 October 1988	10

16 SUPPLEMENTARY NOTATION

17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	Band matrix, inverse eigenvalue problem, Givens rotations

19 ABSTRACT (Continue on reverse if necessary and identify by block number)

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT ☐ DTIC USERS	21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a NAME OF RESPONSIBLE INDIVIDUAL William Gragg	22b TELEPHONE (Include Area Code) (408) 646-2194 / 22c OFFICE SYMBOL 53Gr

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete

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A NOTE ON AN INVERSE EIGENPROBLEM FOR BAND MATRICES†

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Abstract

We present an efficient rotation pattern that can be used in the construction of a band matrix from spectral data. The procedure allows for the stable $O(n^2)$ construction of a real symmetric band matrix having specified eigenvalues and first p components of its normalized eigenvectors. The procedure can also be used in the second phase of the construction of a band matrix from the interlacing eigenvalues as described in [1]. Previously presented algorithms for these reductions using elementary orthogonal similarity transformations require $O(n^3)$ arithmetic operations.

Key Words: Band matrix, inverse eigenvalue problem, Givens rotations

AMS subject classification: 65F30

Submitted to SIAM Journal on Matrix Analysis and Applications.

† This research was supported in part by the National Science Foundation under grant DMS-8704196.

* Research supported in part by the Naval Postgraduate School Research Council

1. Introduction.

Let A be a real symmetric $(2p+1)$-band matrix of order n, and let A_k denote the trailing principal submatrix of $A = A_n$ of order k. It is well known that the eigenvalues of A_k interlace those of A_{k+1} for each $k < n$, and moreover, given real numbers $\lambda_j^{(k)}$ $(1 \leq j \leq k,\ n-p \leq k \leq n)$ satisfying

$$\lambda_j^{(k+1)} \leq \lambda_j^{(k)} \leq \lambda_{j+1}^{(k+1)}, \qquad (1)$$

there is a $(2p+1)$-band matrix $A = A_n$ such that the eigenvalues of A_k are $\{\lambda_j^{(k)}\}_{j=1}^k$ for each k. In general, this band matrix is not uniquely determined.

The problem of constructing a band matrix from the interlacing eigenvalues (1) is considered in [2] and [1]. A survey of this problem and some related inverse eigenvalue problems is given in [3]. In [2] the interlacing eigenvalues are used to determine the first p components of normalized eigenvectors for A, and the remaining components of the eigenvectors (and hence A) are constructed using a block Lanczos process. In [1] a matrix of bordered structure (where the trailing principal submatrix of order p is diagonal) is constructed that satisfies the required spectral conditions. Householder transformations that preserve the eigenvalues of the trailing submatrices are then applied to reduce this bordered matrix to band form. This reduction procedure uses $O(n^3)$ arithmetic operations.

In this note we present an efficient rotation pattern that provides a stable $O(n^2)$ procedure which can be used in the second step (the reduction step) of either of the above methods. This algorithm provides a solution to the open problem posed in [3, p.615], and can be considered as the generalization to band matrices of Rutishauser's procedure for the construction of Jacobi matrices from spectral data presented in [4].

2. The Algorithm

The reduction step in [2] can be described as follows. Given $\{\lambda_j\}_{j=1}^n$ and an $n \times p$ matrix Q_1 with orthogonal columns, construct a $(2p+1)$-band matrix having eigenvalues λ_j and such that Q_1^T forms the first p rows of the (orthogonal) eigenvector matrix for A. This reduction can be performed using a sequence of orthogonal similarity transformations whose composition results in an orthogonal transformation Q such that

$$\begin{bmatrix} I_p & 0 \\ 0 & Q^T \end{bmatrix} \begin{bmatrix} X & Q_1^T \\ Q_1 & \Lambda \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} X & I_p & 0 \\ I_p & & \\ 0 & & A \end{bmatrix} \tag{2}$$

is a $(2p+1)$-band matrix of order $n+p$. The trailing principal submatrix $A=A_n$ then satisfies the required spectral conditions. (The matrix X is arbitrary and remains unchanged).

In the algorithm given in [1], a matrix of the bordered form

$$B = \begin{bmatrix} B_0 & B_1^T \\ B_1 & D \end{bmatrix}, \tag{3}$$

where D is a diagonal matrix of order $n-p$, is constructed such that the trailing principal submatrices of orders $n-p$ through n of B have prescribed eigenvalues. Householder transformations that do not involve the first p coordinate axes are then used to transform B to a $(2p+1)$-band matrix A while preserving the eigenvalues of the trailing principal submatrices. In particular, the composition of these Householder transformations yields an orthogonal matrix U of order $n-p$ such that

$$A = \begin{bmatrix} I_p & 0 \\ 0 & U^T \end{bmatrix} \begin{bmatrix} B_0 & B_1^T \\ B_1 & D \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & U \end{bmatrix}$$

is a $(2p+1)$-band matrix of order n. Thus, the reduction of the matrices in (2) and (4) is essentially the same problem. We now describe our efficient rotation pattern in terms of the reduction of a matrix in the bordered form (3).

The efficient reduction to band form is obtained by performing rotations to introduce appropriate zeros in B row-by-row beginning at row $p+2$, in such a way that *the intermediate matrices remain sparse.* In contrast, a Householder transformation to introduce zeros in the first column of the matrix will result in a full matrix, and the subsequent Householder transformations must be performed on full matrices.

Let $R(A,j,k,l) = G\,A\,G^T$, where G is the elementary Givens rotation in the (j,k)-plane that annihilates a_{kl}. Thus, G is the identity matrix if $a_{kl}=0$. If $a_{kl}\neq 0$ then G is the identity matrix apart from the 2×2 submatrix formed from rows and columns j and k, which is given by

$$G\begin{bmatrix} j, & k \\ j, & k \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix},$$

where $c := a_{jl}/\sqrt{a_{jl}^2 + a_{kl}^2}$ and $s := a_{kl}/\sqrt{a_{jl}^2 + a_{kl}^2}$. Our algorithm for reducing the bordered matrix to band form is then given as follows.

Algorithm.

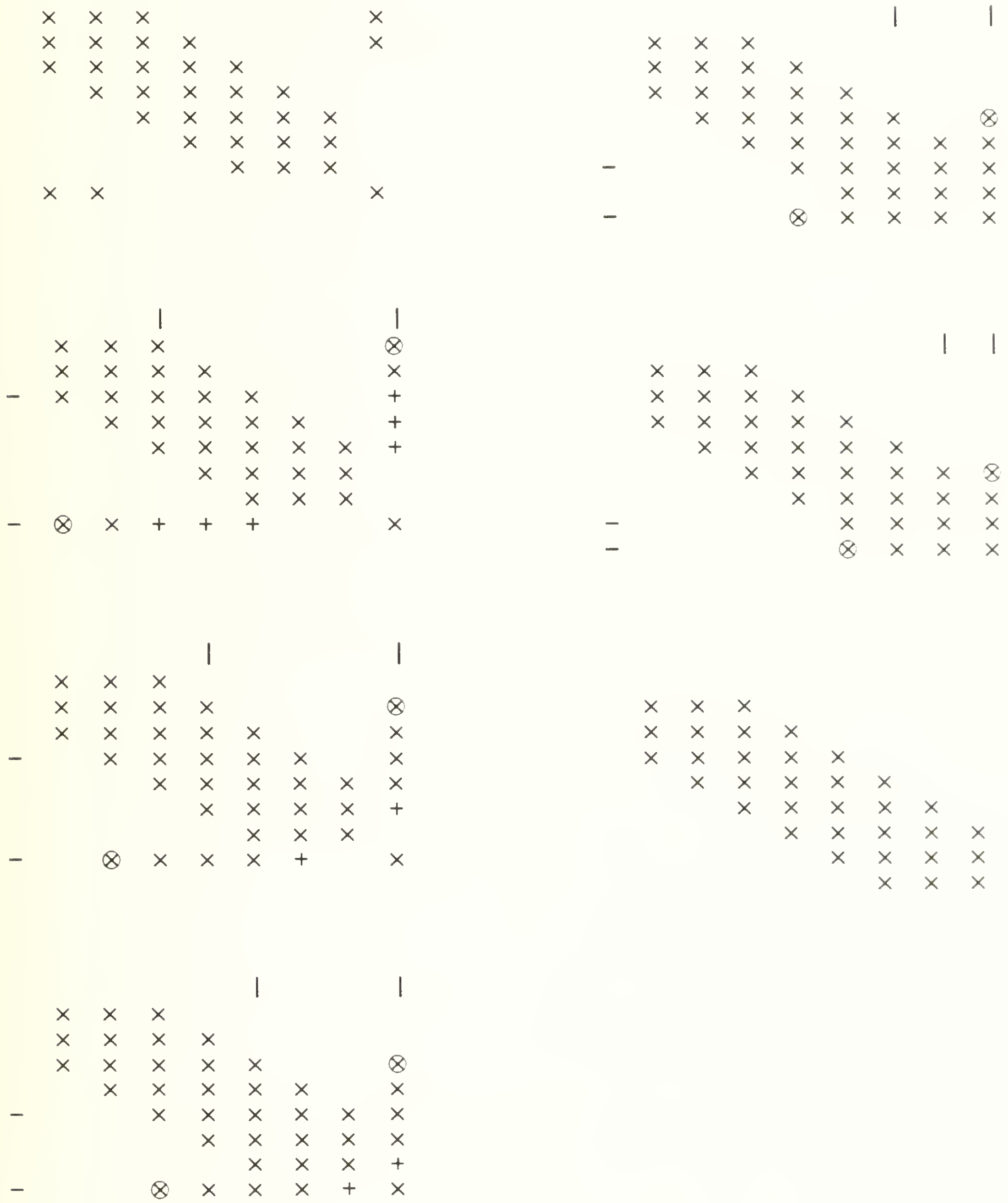
for $k = p+2, \ldots, n$
- **for** $j = p+1, \ldots, k-1$
  - $A := R(A, j, k, j-p)$

To see how the sparsity is preserved, consider the example in Figure 1. There $n = 8$, $p = 2$, and the necessary zeros have already been introduced in rows 4 through 7. Nonzero entries are represented by ×, a Givens rotation is performed in the indicated planes to annihilate the circled entry, and the symbol + indicates the "fill in" (i.e., the additional nonzero entries) introduced by the rotation. The first rotation, in the (3,8) plane, annihilates $a_{8,1}$ and creates $p+1 = 3$ additional nonzero entries. (We count a_{ij} and a_{ji} as one element.) The successive rotations introduce at most one additional nonzero element each, so there are at most $2p+1 = 5$ nonzero entries on the 8th row at any time. We can therefore perform each elementary similarity transformations on A in $O(p)$ arithmetic work. Thus the amount of computation required by the reduction is $O(pn^2)$.

Our algorithm for the reduction of a bordered matrix to band form is explicitly given below. This description involves only the lower-triangular part of the symmetric matrix A.

FIGURE 1.

Rotations are performed in coordinate planes (3,8), (4,8), (5,8), (6,8) and (7,8) to introduce the appropriate zeros in the eighth row.



Algorithm.

Input: a symmetric matrix $A = [a_{j,k}]_{j,k=1}^{n}$ whose trailing principal submatrix of order $n-p$ is diagonal.

Output: a symmetric $(2p+1)$-band matrix A whose trailing principal submatrices of orders $n-p$ through n are orthogonally similar with those of the input matrix.

for $k = p+2, \ldots, n$
- **for** $j = p+1, \ldots, k-1$
  - **if** $a_{k,j-p} \neq 0$ **then**
    - $\rho := \sqrt{a_{j,j-p}^2 + a_{k,j-p}^2}$;
    - $c := a_{j,j-p}/\rho$; $s := a_{k,j-p}/\rho$;
    - $a_{j,j-p} := \rho$; $a_{k,j-p} := 0$;
    - **for** $i = p-1, p-2, \ldots, 1$
      - $\begin{bmatrix} a_{j,j-i} \\ a_{k,j-i} \end{bmatrix} := \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} a_{j,j-i} \\ a_{k,j-i} \end{bmatrix}$
    - **for** $i = j+1, j+2, \ldots, \min\{j+p, k-1\}$
      - $\begin{bmatrix} a_{i,j} \\ a_{k,i} \end{bmatrix} := \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} a_{i,j} \\ a_{k,i} \end{bmatrix}$
    - $u := a_{j,j}$, $v := a_{k,k}$, $w := a_{k,j}$
    - $a_{j,j} := c^2u + s^2v + 2csw$; $a_{k,k} := c^2v + s^2u - 2csw$;
    - $a_{k,j} := cs(v-u) + (c^2 - s^2)w$.

3. Numerical results.

Numerical experiments verify that our efficient rotation pattern produces accurate results in lower order work than the Householder reduction technique. These experiments were performed on the VAX 11/750 at Northern Illinois University.

The following experiment was performed. The method of [1] was used to create a bordered matrix whose trailing principal matrices of order $n-p$ through n have specified eigenvalues. This matrix was then reduced to $(2p+1)$-band form using

I. the Householder reduction procedure of [1];

II. our efficient rotation pattern.

We calculated the average and maximum absolute error among the assigned eigenvalues of the trailing principal submatrices of orders $n-p$ through n. The results displayed in Table 1 were obtained by assigning the eigenvalues of A_k, $n-p \leq k \leq n$, to be the integers $2j+(n-k-1)$, $1 \leq j \leq k$. Experiments were carried out on a variety of other problems with similar results.

Table 1. Errors in eigenvalues.

		average error		maximum error	
n	p	I	II	I	II
10	2	0.1236e-05	0.5762e-06	0.9537e-05	0.1907e-05
20	2	0.3699e-05	0.2226e-05	0.2289e-04	0.9537e-05
50	2	0.9784e-05	0.1210e-04	0.5341e-04	0.4578e-04
10	4	0.1283e-05	0.5470e-06	0.5722e-05	0.1907e-05
20	4	0.2068e-05	0.2948e-05	0.1335e-04	0.1144e-04
50	4	0.1106e-04	0.1199e-04	0.4578e-04	0.6866e-04
10	6	0.7600e-06	0.4705e-06	0.2861e-05	0.1907e-05
20	6	0.2815e-05	0.3268e-05	0.7629e-05	0.1144e-04
50	6	0.1189e-04	0.2058e-04	0.6866e-04	0.6104e-04

Tables 2a and 2b show average average CPU times used by each reduction scheme for various values of n and p. Table 2c shows the corresponding ratios of the time used by the Householder reduction to that of our rotation pattern. These ratios represent the speedup factors of Algorithm II relative to Algorithm I. Note that for fixed n, the amount of computation required by Algorithm I decreases as p increases, while that of Algorithm II is often increasing as a function of p when p is small. These results show that our rotation pattern is consistently more efficient than the Householder reduction technique. The relative efficiency of the rotation pattern generally increases as n increases and decreases as p increases.

Table 2a. Average timings for Algorithm I (CPU seconds).

n / p	10	20	30	40	50	100	200
1	0.029	0.182	0.550	1.231	2.342	17.858	140.070
2	0.023	0.163	0.534	1.199	2.286	17.632	139.693
5	0.013	0.131	0.456	1.081	2.119	17.127	137.837
10		0.072	0.327	0.868	1.796	15.852	133.120
20			0.117	0.476	1.178	13.503	123.227

Table 2b. Average timings for Algorithm II (CPU seconds).

n / p	10	20	30	40	50	100	200
1	0.022	0.087	0.207	0.381	0.596	2.493	10.273
2	0.018	0.099	0.244	0.453	0.734	3.112	13.037
5	0.009	0.103	0.302	0.618	1.044	4.807	20.757
10		0.063	0.287	0.692	1.275	6.937	32.130
20			0.104	0.451	1.110	9.250	50.007

Table 2c. Ratios of CPU times.							
n	10	20	30	40	50	100	200
p							
1	1.346	2.096	2.661	3.232	3.931	7.162	13.634
2	1.333	1.639	2.188	2.645	3.114	5.666	10.715
5	1.364	1.266	1.511	1.748	2.030	3.563	6.641
10		1.147	1.136	1.253	1.408	2.285	4.143
20			1.128	1.055	1.062	1.460	2.464

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